On DLA's η

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SUMMARY

In his pioneering paper on seismic anisotropy in a layered earth, Anderson (1961) introduced a parameter often referred in global seismology as η without providing any reasoning. This note hopes to clarify the significance of η in the context of the dependence of bodywave velocities in a transversely isotropic system on the angle of incidence, and also its relation with the other well-known anisotropic parameters introduced by Thomsen (1986).

Key words: Seismic anisotropy, transverse isotropy, radial anisotropy.

Introduction

To describe a radially anisotropy (transversely isotropy with a vertical symmetry axis, VTI) system, we employ the Love's original notation (Love, 1927), where stress and strain tensors are related by

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} A & H & F & & & \\ H & A & F & & & \\ F & F & C & & & \\ & & & L & & \\ & & & & L & \\ & & & & & N \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix} \tag{1}$$

where $H = A - 2N$. There are five independent parameters, $A,\ C,\ F,\ L,\ N$, to describe this system, while there are two, $\lambda,\ \mu$, for the isotropic case, for which $A = C = \lambda + 2\mu$, $F = \lambda$, $L = N = \mu$. For convenience, Anderson (1961) introduced the following "anisotropy factors":

$$\varphi = C/A = \alpha_V^2/\alpha_H^2 \tag{2}$$

$$\xi = (A-H)/2L = N/L = \beta_H^2/\beta_V^2 \tag{3}$$

$$\eta = (A-2L)/F, \tag{4}$$

which are all equal to 1 for isotropic case ($\alpha_V = \sqrt{C/\rho}$, $\alpha_H = \sqrt{A/\rho}$, $\beta_H = \sqrt{N/\rho}$, $\beta_V = \sqrt{L/\rho}$, where ρ gives the density.).

While both φ and ξ have simple meanings (degree of anisotropy in P- and S-wave, respectively), the physical meaning of η is not so trivial. Takeuchi and Saito (1972), in their monograph on seismic surface waves, reversed the order of the denominator and numerator in the definition of η as,

$$\eta \;=\; F/(A-2L), \tag{5}$$

without commenting on the physical meaning either. As the expression of Takeuchi and Saito (1972) is now commonly used in the global seismological community, we will use this notation and denote it as $\eta_{DLA} = F/(A-2L)$ in the following. In his text book, Anderson (1989) called this η_{DLA} "the fifth parameter required to fully describe transverse isotropy". In Dziewonski and Anderson (1981), by showing examples, the effect of η_{DLA} on the incident angle dependence of the phase velocity of P and S waves is discussed, and we generally think that η_{DLA} controls, to some extent, the incidence angle dependence of those bodywaves, as well as those of Rayleigh waves.

The purpose of this short note is provide simple theoretical background to how η_{DLA} affects the bodywave propagation.

Incidence angle dependence of bodywaves

By solving an eigenvalue problem of an appropriate Christoffel matrix, the incident angle, θ dependence of bodywave phase velocities can be obtained as

$$\rho v_P^2(\theta) \;=\; \frac{(L+C)+(A-C)\sin^2\theta+\sqrt{S}}{2} \tag{6}$$

$$\rho v_{SV}^2(\theta) \;=\; \frac{(L+C)+(A-C)\sin^2\theta-\sqrt{S}}{2} \tag{7}$$

$$\rho v_{SH}^2(\theta) \;=\; L+(N-L)\sin^2\theta, \tag{8}$$

where v_P, v_{SV}, and v_{SH} denote phase velocities of pseudo- P, SV and SH waves respectively, and

$$S \;=\; \{(A-L)\sin^2\theta-(C-L)\cos^2\theta\}^2+(F+L)^2\sin^2 2\theta \tag{9}$$

$$\;=\; \{(A-L)\sin^2\theta+(C-L)\cos^2\theta\}^2+\{(F+L)^2-(C-L)(A-L)\}\sin^2 2\theta \tag{10}$$

$$\;=\; \{(C-L)+(A-C)\sin^2\theta\}^2+\{(F+L)^2-(C-L)(A-L)\}\sin^2 2\theta \tag{11}$$

$$\;=\; (C-L)^2+(A-C)(A+C-2L)\sin^2\theta+\{(F+L)^2-(\frac{A+C}{2}-L)^2\}\sin^2 2\theta. \tag{12}$$

When the condition

$$(F+L)^2=(C-L)(A-L) \tag{13}$$

is satisfied, equation (11) will be $S=\{(C-L)+(A-C)\sin^2\theta\}^2$, and

$$\rho v_P^2(\theta) \;=C+(A-C)\sin^2\theta \tag{14}$$

$$\rho v_{SV}^2(\theta)=L \tag{15}$$

$$\rho v_{SH}^2(\theta)=L+(N-L)\sin^2\theta. \tag{16}$$

The condition (13) is called by Thomsen (1986) the elliptic condition, since, in the absence of the $\sin^2 2\theta$ term, the forms of the wave velocity surfaces as a function of incidence angle θ are elliptical with only a $\sin^2\theta$ dependence. When condition (13) is not satisfied the presence of the $\sin^2 2\theta$ term means that the wavesurfaces can be either convex or concave. (The convexity/concavity of the P velocity is in the opposite sense to that of the SV velocity. This is an explicit consequence of the presence of the $\sqrt{S}$ term in (6) and (7) with opposite signs.)

Thus if we were to introduce an additional parameter to characterize the incidence angle dependence of bodywaves, one reasonable choice may be

$$\eta_\kappa \;=\; \frac{F+L}{(A-L)^{1/2}(C-L)^{1/2}}, \tag{17}$$

and $\eta_\kappa = 1$ for the isotropic case.

Further considering

$$(A-L)(C-L) \;=\; \left(\frac{A+C}{2} - L\right)^2 \;-\; \left(\frac{A-C}{2}\right)^2,$$

$$\eta_{\kappa'} \;=\; \frac{F+L}{\frac{A+C}{2} - L} \tag{18}$$

may be another possibility that might make sense by looking at equation (12).

One of the good points of η_{DLA} is that it is simple and depends on just A and not C. Assuming P-wave anisotropy is small, if we substitute $\frac{A+C}{2}$ in (18) by A, we get

$$\eta_{\kappa''} \;=\; \frac{F+L}{A-L} \tag{19}$$

It is instructive how these parameters (η's) behave when both P- and S-wave anisotropy is absent (i.e., $A = C$ and $L = N$). When these conditions are satisfied,

$$\begin{aligned}
\rho v_P^2(\theta) &= \frac{(L+A)+\sqrt{S}}{2} \\
\rho v_{SV}^2(\theta) &= \frac{(L+A)-\sqrt{S}}{2} \\
\rho v_{SH}^2(\theta) &= L,
\end{aligned}$$

and

$$S = \{(A-L)\}^2 + \{(F+L)^2 - (A-L)^2\}\sin^2 2\theta,$$

and $\sin^2\theta$ dependence disappears. In this case, η_κ, $\eta_{\kappa'}$, and $\eta_{\kappa''}$ reduce to the same form. Also, it is easy to see that $\eta_{DLA} = 1$ gives the elliptic condition, and so in this sense, $\eta_{DLA} - 1$ becomes a measure of a departure from the elliptic condition to dictate the convex/concave pattern.

For more general case, $\chi = \eta_{DLA} - 1$ is small for weak anisotropy,

$$\chi = \eta_{DLA} - 1 \;=\; \frac{F - A + 2L}{A - 2L}. \tag{20}$$

Similarly

$$\chi'' = \eta_{\kappa''} - 1 \;=\; \frac{F - A + 2L}{A - L} \;=\; \chi \times \frac{A - 2L}{A - L}, \tag{21}$$

and as long as $A - L > A - 2L > 0$ is satisfied, χ'' has the same sign as χ, and $\chi > \chi''$, indicating χ'' is also small. So in this respect, if anisotropy is weak (especially in P), η_{DLA} might be a good proxy for η_κ whose departure from unity provides a measures of the deviation from elliptic anisotropy and dictates the convex/concave pattern of the incidence angle dependence of v_P and v_{SV}.

Thomsen's parameters

Thomsen (1986) introduced three parameters for VTI system, now referred to as Thomsen's parameters, and they are defined as

$$
\begin{aligned}
\varepsilon &= \frac{A-C}{2C} = \frac{1}{2}(\varphi^{-1}-1) && (22)\\
\gamma &= \frac{N-L}{2L} = \frac{1}{2}(\xi-1) && (23)\\
\delta &= \frac{(F+L)^2-(C-L)^2}{2C(C-L)}, && (24)
\end{aligned}
$$

which are all small for weak anisotropy. While ε and γ are directly related to φ and ξ respectively as shown above and thus to P- and S-wave anisotropy, δ was introduced such that it dominates v_P in the case of near vertical incidence as in reflection profiling.

Considering that $\delta = \varepsilon$ is their condition for elliptical anisotropy, examination of $\varepsilon - \delta$ leads to

$$
\begin{aligned}
\varepsilon - \delta &= \frac{A-C}{2C} - \frac{(F+L)^2-(C-L)^2}{2C(C-L)} && (25)\\
&= \frac{(A-L)(C-L)-(F+L)^2}{2C(C-L)} && (26)\\
&= \left(1-\eta_\kappa^2\right)\frac{A-L}{2C}, && (27)
\end{aligned}
$$

and we now see the connection between Thomsen's δ and η_κ introduced here. If η_{DLA} were a proxy of η_κ for weak anisotropy, we might be able to say that a connection between η_{DLA} and Thomsen's δ is established.

For weak anisotropy, the incidence angle dependence of bodywaves are, according to Thomsen (1986),

$$
\begin{aligned}
v_P(\theta) &= \alpha_H\left(1+\delta\sin^2\theta\cos^2\theta+\varepsilon\sin^4\theta\right) && (28)\\
v_{SV}(\theta) &= \beta_V\left[1+\frac{\alpha_H^2}{\beta_V^2}(\varepsilon-\delta)\sin^2\theta\cos^2\theta\right] && (29)\\
v_{SH}(\theta) &= \beta_V(1+\gamma\sin^2\theta), && (30)
\end{aligned}
$$

and when the elliptic condition is satisfied

$$
\begin{aligned}
v_P(\theta) &= \alpha_H(1+\varepsilon\sin^2\theta)\\
v_{SV}(\theta) &= \beta_V\\
v_{SH}(\theta) &= \beta_V(1+\gamma\sin^2\theta),
\end{aligned}
$$

which show simple incidence angle dependences.

(28)(29)(30) may be expressed in terms of 2θ and 4θ to make the incidence angle dependence more explicit:

$$
\begin{aligned}
v_P(\theta) &= \alpha_H\left[1+\frac{\varepsilon}{2}(1-\cos 2\theta)-\frac{\omega}{2}(1-\cos 4\theta)\right] && (31)\\
v_{SV}(\theta) &= \beta_V\left[1+\frac{\alpha_H^2}{\beta_V^2}\frac{\omega}{2}(1-\cos 4\theta)\right] && (32)\\
v_{SH}(\theta) &= \beta_V\left[1+\frac{\gamma}{2}(1-\cos 2\theta)\right], && (33)
\end{aligned}
$$

where $\omega = (\varepsilon-\delta)/4$ is introduced. These equations show that $(\varepsilon-\delta)$ dictates the convex/concave nature (i.e, $\cos 4\theta$ dependence) of v_P and v_{SV}.

η_{DLA} and η_κ for weakly anisotropic models

To finish up this short note, we compare distributions of η-related parameters for some of weakly anisotropic cases.

Millefeuille (isotropic layers) case

In the first example, we present a series of VTI models constructed by the Backus averaging (Backus, 1962) of a stack of two kinds of homogeneous isotropic layers: soft layers embedded in a background solid matrix (e.g., Kawakatsu et al., 2009). We parameterize (i) the proportional reduction of rigidity of soft layers to the background by a ($0 \leq a \leq 1$), (ii) the proportional reduction of the bulk modulus by $a/2$, and (iii) the volume fraction of soft layers by f ($0 \leq f \leq 1$). Both a and f are varied in intervals of 0.05. Figure 1(a) compares η_κ with η_{DLA} (blue circles) or $\eta_{\kappa'}$ (magenta crosses). While η_κ and $\eta_{\kappa'}$ give almost the same values, η_{DLA}

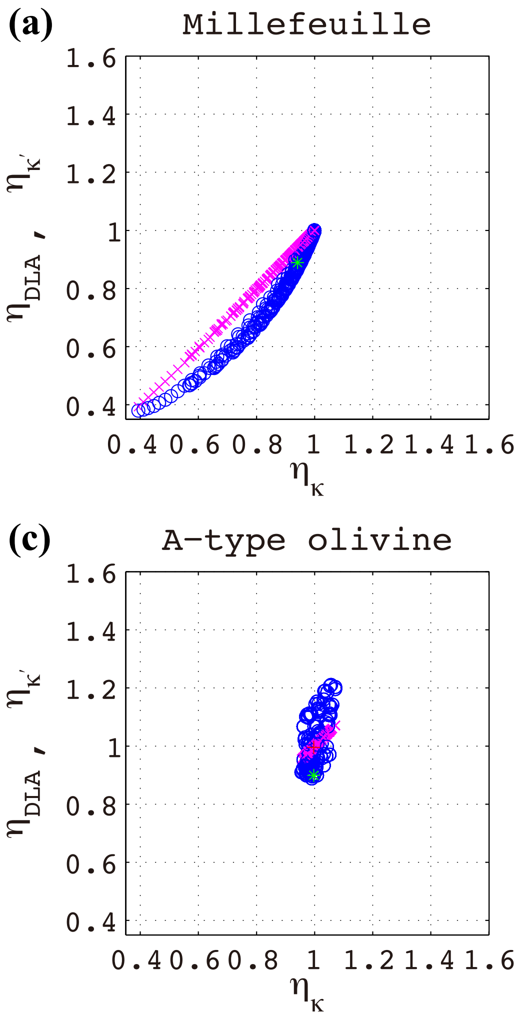

Figure 1: Comparison of η-related parameters for various weakly anisotropic models. (a) Millefeuille case, (b) general TI case, and (c) rotated A-type olivine case. Green asterisks correspond to η_κ *vs.* η_{DLA} for (a) $a = 0.9$, $f = 0.01$, (b) peak-to-peak anisotropy for both P and S waves is 1.5% with $\eta_{DLA} = 0.9, 1.0$, and (c) the A-type olivine fabric case whose fast axis lies in the horizontal plane, all for which examples of incident angle dependency of bodywaves are shown in Figure 2.

gives slightly smaller values. As $\eta_\kappa \leq 1$ is guaranteed (Berryman, 1979), all values appear generally less than 1. Although η_{DLA} in this case slightly deviates from η_κ, nearly one-to-one correspondence may be observed to make η_{DLA} a reasonable proxy to η_κ.

General case

For a more general case, we construct a series of VTI models which have a maximum of $\pm 5\%$ anisotropy in both $\alpha_{V,H}$ and $\beta_{V,H}$, and $0.5 < \eta_{DLA} < 1.5$ (Figure 1(b)). While η_κ and $\eta_{\kappa'}$ give almost the same values, η_{DLA} deviates significantly from the corresponding η_κ.

A-type olivine case

As a third example, we construct a series of VTI models by azimuthal averaging (Montagner and Nataf, 1986; Montagner and Anderson, 1989) of an arbitrarily rotated A-type olivine fabric (Jung et al., 2006) (Figure 1(c)) (rotation is done with a 30-degree interval for each Euler angle). In a similar way to the preceding cases, η_κ and $\eta_{\kappa'}$ have almost the same values, but η_{DLA} deviates from corresponding η_κ.

Examples of the incidence angle dependence of representative VTI models (denoted by green asterisks in Figure 1) are shown in Figure 2. Note that the convex pattern of v_{SV} velocity occurs when $\eta_\kappa < 1$.

Discussion

The incidence angle dependence of bodywave phase velocities in a radially anisotropic system has not been discussed much in the geophysical literature as it is a difficult effect to observe. In the laboratory, on the other hand, the simple $\sin\theta$ and $\sin 2\theta$ dependence (e.g., (6) and (11)) has been used to obtain the fifth elastic constant from measurement along the angle 45 degrees from the symmetric axes (e.g., Christensen and Crosson, 1968; Anderson, 1966). Song and Kawakatsu (2012, 2013) recently suggested that such incident angle dependency in the Earth may be constrained at subduction zones where the dip of the lithosphere/asthenosphere changes along with the subduction, affecting the effective incidence angle of teleseismic bodywaves to the system. If such analyses can be made generally, the new parameter η_κ (or $\eta_{\kappa'}$) might be a useful tool in global seismology to characterize VTI (radially anisotropic) systems. How η-related parameters might be constrained from Rayleigh wave dispersion needs also to be understood (e.g., Anderson, 1966).

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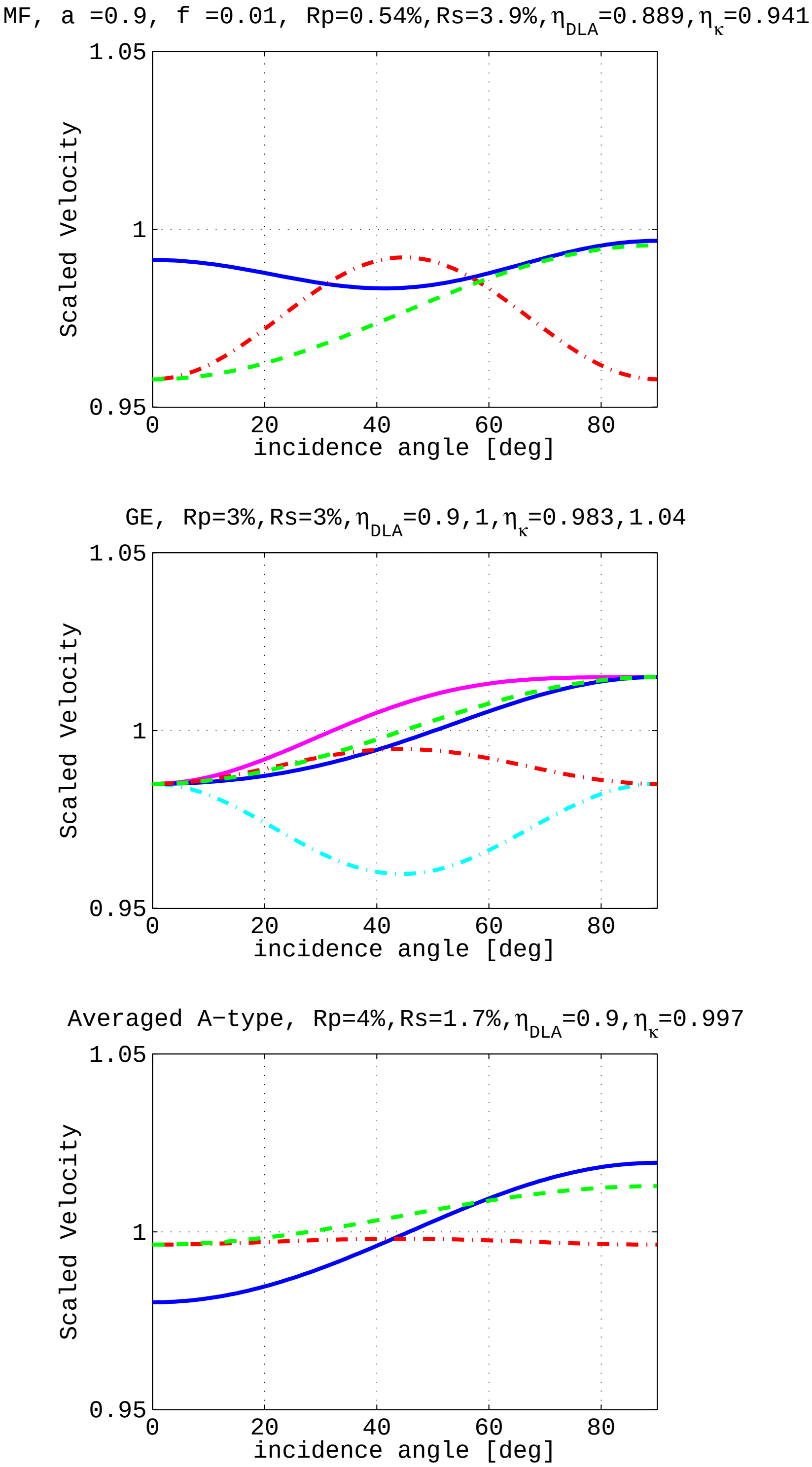

Figure 2: Examples of the incidence angle dependency of bodywaves for the VTI models represented by asterisks in Figure 1. Blue (and magenta in middle) solid lines, red (and cyan) dash-dot lines, and green dashed lines are respectively for v_P, v_{SV}, and v_{SH}. Phase velocities are scaled by those of corresponding reference isotropic models. (Top), (middle), and (bottom) correspond to the models in (a), (b) and (c) in Figure 1. In the middle panel, v_P and v_{SV} shown by magenta and cyan lines are for $\eta_{DLA} = 1$, $\eta_\kappa = 1.04$ case, and v_{SH} behaves the same for two cases.