THE DISPERSION OF SURFACE WAVES ON MULTILAYERED MEDIA*

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ABSTRACT

A matrix formalism developed by W. T. Thomson is used to obtain the phase velocity dispersion equations for elastic surface waves of Rayleigh and Love type on multilayered solid media. The method is used to compute phase and group velocities of Rayleigh waves for two assumed three-layer models and one two-layer model of the earth's crust in the continents. The computed group velocity curves are compared with published values of the group velocities at various frequencies of Rayleigh waves over continental paths. The scatter of the observed values is larger than the difference between the three computed curves. It is believed that not all of this scatter is due to observational errors, but probably represents a real horizontal heterogeneity of the continental crusts.

Introduction

In the usual treatment[1] of the dispersion of surface waves of Rayleigh type on a layered medium, the dependence of phase velocity upon wave length is expressed by the vanishing of a certain determinant whose elements are functions of the phase velocity, the wave length, and the densities and elastic moduli of the various layers. If there are n layers (including the last, which is assumed to be semi-infinite), there are $4n - 2$ boundary conditions to be satisfied: continuity of two displacement components and two stress components at each interface, and vanishing of two stress components at the free surface. These lead to $4n - 2$ homogeneous simultaneous equations to determine an equal number of unknown constants. A solution exists only if the determinant of the coefficients vanishes. Although many of the elements of this $4n - 2$ rowed determinant are zero, the computational labor involved in determining the roots is so formidable that no attempt appears to have been made to treat cases of more than two layers by this method.

In the present paper the problem is reformulated in terms of matrices, following a method introduced by W. T. Thomson.[2] Although this may be regarded as no more than a change in notation, the matrix notation itself suggests a systematic computational procedure that makes it possible to handle at least a three-layer case on an ordinary desk calculator without an unreasonable expenditure of time.

In the case of Love waves there are only two boundary conditions to be satisfied at each interface, and the three-layer problem can be treated by straightforward methods without excessive algebraic manipulation. However, for more than three layers the matrix method may be advantageous in this case also.

In order to simplify the discussion as far as possible, we shall consider solutions of the elastic equations of motion in the form of plane waves rather than attempt to treat the more complex case of waves diverging from a point source. So far as the dispersion function is concerned, this involves no loss of generality, since the point-

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[1] K. Sezawa, "Dispersion of Elastic Waves Propagated on the Surface of Stratified Bodies and on Curved Surfaces," *Bull. Earthq. Res. Inst. Tokyo*, 3: 1–18 (1927), esp. p. 16; A. W. Lee, "The Effect of Geological Structure upon Microseismic Disturbance," *Mon. Not. Roy. Astron. Soc., Geophys. Suppl.*, 3: 83–105 (1932); C. Y. Fu, "Rayleigh Waves in a Superficial Layer," *Geophysics*, 11: 10–23 (1946).

[2] W. T. Thomson, "Transmission of Elastic Waves through a Stratified Solid Medium," *Jour. Appl. Phys.*, 21: 89 (1950).

source solution may be developed by integration of plane-wave solutions. The plane-wave solution will not, of course, determine the way in which the relative excitation of the various possible normal modes of propagation varies with depth of source and with frequency.

Matrix Formulation of the Problem for Rayleigh Waves

Since there is an easily corrected error in Thomson's development, we shall repeat his derivation in some detail, with certain minor changes in notation that will exhibit the basic symmetry of the final expressions somewhat more clearly.

We consider plane waves of angular frequency p and horizontal phase velocity c propagated in a semi-infinite medium made up of n parallel, homogeneous, isotropic

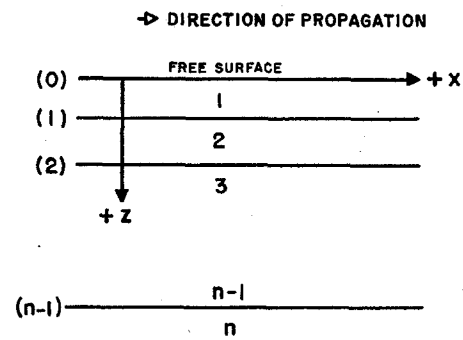

Fig. 1. Direction of axes and numbering of layers and interfaces.

layers. For the present, all layers will be assumed to be solid; the case of a fluid layer will be considered later. The x axis is taken parallel to the layers with the positive sense in the direction of propagation. The positive z axis is taken as directed into the medium. The various layers and interfaces are numbered away from the free surface, as shown in figure 1. We confine our attention to waves of Rayleigh type, by which we mean that there is no displacement in the y direction and that the amplitude diminishes exponentially in the +z direction in the semi-infinite layer.

For the mth layer let

$$\begin{aligned}
\rho_m &= \text{density} \\
d_m &= \text{thickness} \\
\lambda_m, \mu_m &= \text{Lamé elastic constants} \\
\alpha_m = [(\lambda_m + 2\mu_m)/\rho_m]^{1/2} &= \text{velocity of propagagion of dilatational waves} \\
\beta_m = [\mu_m/\rho_m]^{1/2} &= \text{velocity of propagation of rotational waves} \\
k = p/c &= 2\pi/\text{wave length (horizontal)} \\
r_{\alpha m} &= \begin{cases} +[(c/\alpha_m)^2 - 1]^{1/2} & c > \alpha_m \\ -i[1 - (c/\alpha_m)^2]^{1/2} & c < \alpha_m \end{cases} \\
r_{\beta m} &= \begin{cases} +[(c/\beta_m)^2 - 1]^{1/2} & c > \beta_m \\ -i[1 - (c/\beta_m)^2]^{1/2} & c < \beta_m \end{cases} \\
\gamma_m &= 2(\beta_m/c)^2 \\
u, w &= \text{displacement components in x and z directions} \\
\sigma = Z_z &= \text{normal stress} \\
\tau = X_z &= \text{tangential stress}
\end{aligned}$$

Then, as is well known, periodic solutions of the elastic equations of motion for the m^{th} layer may be found by combining dilatational wave solutions.

$$\Delta_m = (\partial u/\partial x) + (\partial w/\partial z) = \exp[i(pt - kx)][\Delta_m' \exp(-ikr_{\alpha m}z) + \Delta_m'' \exp(ikr_{\alpha m}z)] \tag{2.1}$$

With rotational wave solutions,

$$\omega_m = (1/2)[(\partial u/\partial z) - (\partial w/\partial x)] = \exp[i(pt - kx)][\omega_m' \exp(-ikr_{\beta m}z) + \omega_m'' \exp(ikr_{\beta m}z)] \tag{2.2}$$

where Δ_m', Δ_m'', ω_m' and ω_m'' are constants. With the sign conventions defined above, the term in Δ_m' represents a plane wave whose direction of propagation makes an angle $\cot^{-1} r_{\alpha m}$ with the +z direction when $r_{\alpha m}$ is real, and a wave propagated in the +x direction with amplitude diminishing exponentially in the +z direction when $r_{\alpha m}$ is imaginary. Similarly, the term in Δ_m'' represents a plane wave making the same angle with the −z direction when $r_{\alpha m}$ is real and a wave propagated in the +x direction with amplitude increasing exponentially in the +z direction when $r_{\alpha m}$ is imaginary. The same remarks apply to the terms in ω_m' and ω_m'' with $r_{\beta m}$ substituted for $r_{\alpha m}$.

The displacements and the pertinent stress components corresponding to the dilatation and rotation given by (2.1) and (2.2) are,

$$u = -(\alpha_m/p)^2(\partial\Delta_m/\partial x) - 2(\beta_m/p)^2(\partial\omega_m/\partial z) \tag{2.3}$$

$$w = -(\alpha_m/p)^2(\partial\Delta_m/\partial z) + 2(\beta_m/p)^2(\partial\omega_m/\partial x) \tag{2.4}$$

$$\sigma = \rho_m[\alpha_m^2\Delta_m + 2\beta_m^2\{(\alpha_m/p)^2(\partial^2\Delta_m/\partial x^2) + 2(\beta_m/p)(\partial^2\omega_m/\partial x\partial z)\}] \tag{2.5}$$

$$\tau = 2\rho_m\beta_m^2[-(\alpha_m/p)^2(\partial^2\Delta_m/\partial x\partial z) + (\beta_m/p)^2\{(\partial^2\omega_m/\partial x^2) - (\partial^2\omega_m/\partial z^2)\}] \tag{2.6}$$

The boundary conditions to be met at an interface between two layers are that these four quantities shall be continuous. Continuity of the displacements is assured if the corresponding velocity components $\dot{u}$ and $\dot{w}$ are made continuous and, since c is the same in all layers, we may take the dimensionless quantities $\dot{u}/c$ and $\dot{w}/c$ to be continuous. Substituting the expressions (2.1) and (2.2) in equations (2.3) to (2.6) and expressing the exponential functions of $ikrz$ in trigonometric form, we find

$$\begin{aligned}\dot{u}/c = &-(\alpha_m/c)^2[(\Delta_m' + \Delta_m'')\cos kr_{\alpha m}z - i(\Delta_m' - \Delta_m'')\sin kr_{\alpha m}z]\\ &-\gamma_m r_{\beta m}[(\omega_m' - \omega_m'')\cos kr_{\beta m}z - i(\omega_m' + \omega_m'')\sin kr_{\beta m}z]\end{aligned} \tag{2.7}$$

$$\begin{aligned}\dot{w}/c = &-(\alpha_m/c)^2 r_{\alpha m}[-i(\Delta_m' + \Delta_m'')\sin kr_{\alpha m}z + (\Delta_m' - \Delta_m'')\cos kr_{\alpha m}z]\\ &+\gamma_m[-i(\omega_m' - \omega_m'')\sin kr_{\beta m}z + (\omega_m' + \omega_m'')\cos kr_{\beta m}z]\end{aligned} \tag{2.8}$$

$$\sigma = -\rho_m\alpha_m^2(\gamma_m - 1)[(\Delta_m' + \Delta_m'')\cos kr_{\alpha m}z - i(\Delta_m' - \Delta_m'')\sin kr_{\alpha m}z]$$

$$- \rho_m c^2 \gamma_m^2 r_{\beta m}[(\omega_m' - \omega_m'') \cos kr_{\beta m}z - i(\omega_m' + \omega_m'') \sin kr_{\beta m}z] \tag{2.9}$$

$$\tau = \rho_m \alpha_m^2 \gamma_m r_{\alpha m}[-i(\Delta_m' + \Delta_m'') \sin kr_{\alpha m}z + (\Delta_m' - \Delta_m'') \cos kr_{\alpha m}z]$$

$$- \rho_m c^2 \gamma_m(\gamma_m - 1)[-i(\omega_m' - \omega_m'') \sin kr_{\beta m}z + (\omega_m' + \omega_m'') \cos kr_{\beta m}z] \tag{2.10}$$

When any of the r's are imaginary, the trigonometric functions are to be understood as going over into the corresponding hyperbolic functions.

Placing the origin of z at the $(m - 1)^{\text{th}}$ interface, the linear relationship between the values of $\dot{u}/c$, $\dot{w}/c$, σ, and τ at the $(m - 1)^{\text{th}}$ interface and the constants $(\Delta_m' + \Delta_m'')$, $(\Delta_m' - \Delta_m'')$, $(\omega_m' - \omega_m'')$, and $(\omega_m' + \omega_m'')$ may be represented by the transformation

$$(\dot{u}_{m-1}/c,\ \dot{w}_{m-1}/c,\ \sigma_{m-1}\tau_{m-1}) = E_m(\Delta_m' + \Delta_m'',\ \Delta_m' - \Delta_m'',\ \omega_m' - \omega_m'',\ \omega_m' + \omega_m'') \tag{2.11}$$

where E_m is the matrix

$$E_m = \begin{bmatrix} -(\alpha_m/c)^2 & 0 & -\gamma_m r_{\beta m} & 0 \\ 0 & -(\alpha_m/c)^2 r_{\alpha m} & 0 & \gamma_m \\ -\rho_m \alpha_m^2(\gamma_m - 1) & 0 & -\rho_m c^2 \gamma_m r_{\beta m} & 0 \\ 0 & \rho_m \alpha_m^2 \gamma_m r_{\alpha m} & 0 & -\rho_m c^2 \gamma_m(\gamma_m - 1) \end{bmatrix} \tag{2.12}$$

Setting $z = d_m$ in equations (2.7) to (2.10) gives the values of $\dot{u}/c$ etc. at the m^{th} interface in terms of $\Delta_m' + \Delta_m''$ etc.

$$(\dot{u}_m/c,\ \dot{w}_m/c,\ \sigma_m,\ \tau_m) = D_m(\Delta_m' + \Delta_m'',\ \Delta_m' - \Delta_m'',\ \omega_m' - \omega_m'',\ \omega_m' + \omega_m'') \tag{2.13}$$

where D_m is the matrix

$$D_m =$$

$$\begin{bmatrix} -(\alpha_m/c)^2 \cos P_m & i(\alpha_m/c)^2 \sin P_m & -\gamma_m r_{\beta m} \cos Q_m & i\gamma_m r_{\beta m} \sin Q_m \\ i(\alpha_m/c)^2 r_{\alpha m} \sin P_m & -(\alpha_m/c)^2 r_{\alpha m} \cos P_m & -i\gamma_m \sin Q_m & \gamma_m \cos Q_m \\ -\rho_m \alpha_m^2(\gamma_m - 1) \cos P_m & i\rho_m \alpha_m^2(\gamma_m - 1) \sin P_m & -\rho_m c^2 \gamma_m^2 r_{\beta m} \cos Q_m & i\rho_m c^2 \gamma_m^2 r_{\beta m} \sin Q_m \\ -i\rho_m \alpha_m^2 \gamma_m r_{\alpha m} \sin P_m & \rho_m \alpha_m^2 \gamma_m r_{\alpha m} \cos P_m & i\rho_m c^2 \gamma_m(\gamma_m - 1) \sin Q_m & -\rho_m c^2 \gamma_m(\gamma_m - 1) \cos Q_m \end{bmatrix} \tag{2.14}$$

with $P_m = kr_{\alpha m}d_m$ and $Q_m = kr_{\beta m}d_m$.

The constants $\Delta_m' + \Delta_m''$ etc. may be eliminated between equations (2.11) and (2.12), giving a linear relationship between the values of $\dot{u}/c$, $\dot{w}/c$, σ, and τ at the top and bottom of the m^{th} layer that may be expressed symbolically by the equation,

$$(\dot{u}_m/c,\ \dot{w}_m/c,\ \sigma_m,\ \tau_m) = D_m E_m^{-1}(\dot{u}_{m-1}/c,\ \dot{w}_{m-1}/c,\ \sigma_{m-1},\ \tau_{m-1}) \tag{2.15}$$

where E_m^{-1} is the inverse of E_m and is given by

$$E_m^{-1} = \begin{bmatrix} -2(\beta_m/\alpha_m)^2 & 0 & (\rho_m\alpha_m^2)^{-1} & 0 \\ 0 & c^2(\gamma_m-1)/\alpha_m^2 r_{\alpha m} & 0 & (\rho_m\alpha_m^2 r_{\alpha m})^{-1} \\ (\gamma_m - 1)/\gamma_m r_{\beta m} & 0 & -(\rho_m c^2\gamma_m r_{\beta m})^{-1} & 0 \\ 0 & 1 & 0 & (\rho_m c^2\gamma_m)^{-1} \end{bmatrix} \tag{2.16}$$

From equations (2.14) and (2.16) the elements of the matrix product $a_m = D_m E_m^{-1}$ may be computed as follows:

$$(a_m)_{11} = \gamma_m \cos P_m - (\gamma_m - 1) \cos Q_m$$

$$(a_m)_{12} = i[(\gamma_m - 1)r_{\alpha m}^{-1} \sin P_m + \gamma_m r_{\beta m} \sin Q_m]$$

$$(a_m)_{13} = -(\rho_m c^2)^{-1}(\cos P_m - \cos Q_m)$$

$$(a_m)_{14} = i(\rho_m c^2)^{-1}(r_{\alpha m}^{-1} \sin P_m + r_{\beta m} \sin Q_m)$$

$$(a_m)_{21} = -i[\gamma_m r_{\alpha m} \sin P_m + (\gamma_m - 1)r_{\beta m}^{-1} \sin Q_m]$$

$$(a_m)_{22} = -(\gamma_m - 1) \cos P_m + \gamma_m \cos Q_m$$

$$(a_m)_{23} = i(\rho_m c^2)^{-1}(r_{\alpha m} \sin P_m + r_{\beta m}^{-1} \sin Q_m)$$

$$(a_m)_{24} = (a_m)_{13}$$

$$(a_m)_{31} = \rho_m c^2\gamma_m(\gamma_m - 1)(\cos P_m - \cos Q_m)$$

$$(a_m)_{32} = i\rho_m c^2[(\gamma_m - 1)^2 r_{\alpha m}^{-1} \sin P_m + \gamma_m^2 r_{\beta m} \sin Q_m]$$

$$(a_m)_{33} = (a_m)_{22}$$

$$(a_m)_{34} = (a_m)_{12}$$

$$(a_m)_{41} = i\rho_m c^2[\gamma_m^2 r_{\alpha m} \sin P_m + (\gamma_m - 1)^2 r_{\beta m}^{-1} \sin Q_m]$$

$$(a_m)_{42} = (a_m)_{31}$$

$$(a_m)_{43} = (a_m)_{21}$$

$$(a_m)_{44} = (a_m)_{11}$$

Now the boundary conditions require that the values of $\dot{u}/c$, $\dot{w}/c$, σ, and τ computed at the top of the m^{th} layer be the same as the values computed at the bottom of the $(m - 1)^{\text{th}}$ layer. This means that we may write

$$(\dot{u}_m/c,\ \dot{w}_m/c,\ \sigma_m,\ \tau_m) = a_m\, a_{m-1}(\dot{u}_{m-2}/c,\ \dot{w}_{m-2}/c,\ \sigma_{m-2},\ \tau_{m-2}) \tag{2.17}$$

In Thomson's paper, the quantity $\tau/2\mu$ is taken instead of τ as the fourth variable on which the matrix a_m operates. This is merely a change in notation so far as any one layer is concerned, but μ will generally be different in different layers, and it is the shearing stress τ, and not the shearing strain τ/μ, that is continuous across the interface. The iterative procedure indicated by equation (2.17) therefore requires that τ, or a constant multiple of τ, rather than $\tau/2\mu$, be taken as the fourth variable. Thomson's matrices a should then be corrected by multiplying the fourth row by 2μ = ($2G$ in Thomson's notation), and the fourth column by $(2\mu)^{-1}$.

By repeated application of equation (2.17) we have,

$$(\dot{u}_{n-1}/c,\ \dot{w}_{n-1}/c,\ \sigma_{n-1},\ \tau_{n-1}) = a_{n-1}a_{n-2}\cdot\cdot\cdot a_1(\dot{u}_0/c,\ \dot{w}_0/c,\ \sigma_0,\ \tau_0) \tag{2.18}$$

and by application of the inverse of equation (2.11) for the n^{th} layer,

$$(\Delta_n' + \Delta_n'',\ \Delta_n' - \Delta_n'',\ \omega_n' - \omega_n'',\ \omega_n' + \omega_n'') = E_n^{-1}a_{n-1}\,a_{n-2}\cdot\cdot\cdot a_1(\dot{u}_0/c,\ \dot{w}_0/c,\ \sigma_0, \tau_0) \tag{2.19}$$

So far the development has been quite general, and equation (2.19) is equally applicable to surface waves or to waves transmitted through the layered medium. The case with which we are particularly concerned is that in which there are no stresses across the free surface, so that $\sigma_0 = \tau_0 = 0$, and there are no sources at infinity, so that $\Delta_n'' = \omega_n'' = 0$. Writing J for the matrix product $E_n^{-1}a_{n-1}\,a_{n-2}\cdot\cdot\cdot\cdot\cdot\cdot a_1$, equation (2.19) becomes,

$$(\Delta_n',\ \Delta_n',\ \omega_n',\ \omega_n') = J(\dot{u}_0/c,\ \dot{w}_0/c,\ 0,\ 0)$$

or, explicitly,

$$\left.\begin{aligned}
\Delta_n' &= J_{11}\dot{u}_0/c + J_{12}\dot{w}_0/c\\
\Delta_n' &= J_{21}\dot{u}_0/c + J_{22}\dot{w}/c\\
\omega_n' &= J_{31}\dot{u}_0/c + J_{32}\dot{w}_0/c\\
\omega_n' &= J_{41}\dot{u}_0/c + J_{42}\dot{w}_0/c
\end{aligned}\right\} \tag{2.20}$$

By eliminating Δ_n' and ω_n' we have,

$$\frac{\dot{u}_0}{\dot{w}_0} = \frac{J_{22} - J_{12}}{J_{11} - J_{21}} = \frac{J_{42} - J_{32}}{J_{31} - J_{41}} \tag{2.21}$$

Since the elements of the matrix J are functions of the parameters c and k, equation (2.21) provides an implicit relationship between c and k, which is the desired phase velocity dispersion function.

Some General Properties of the Solution

Setting $A = a_{n-1}\,a_{n-2}\cdot\cdot\cdot a_1$ and using equation (2.16) for E_n^{-1}, equation (2.21) may be written in the form

$$-(\dot{u}_0/\dot{w}_0) = K/L = M/N \tag{3.1}$$

where

$$\left.\begin{aligned} K &= \gamma_n r_{\alpha n} A_{12} + (\gamma_n - 1)A_{22} - r_{\alpha n} A_{32}/\rho_n c^2 + A_{42}/\rho_n c^2 \\ L &= \gamma_n r_{\alpha n} A_{11} + (\gamma_n - 1)A_{21} - r_{\alpha n} A_{31}/\rho_n c^2 + A_{41}/\rho_n c^2 \\ M &= -(\gamma_n - 1)A_{12} + \gamma_n r_{\beta n} A_{22} + A_{32}/\rho_n c^2 + r_{\beta n} A_{42}/\rho_n c^2 \\ N &= -(\gamma_n - 1)A_{11} + \gamma_n r_{\beta n} A_{21} + A_{31}/\rho_n c^2 + r_{\beta n} A_{41}/\rho_n c^2 \end{aligned}\right\} \tag{3.2}$$

In the two-layer case, equations (3.1) and (3.2) may be shown to be equivalent to the expressions that have previously been derived by Sezawa, Lee, and others.

In the expressions for the elements of the matrices a_m it will be observed that the quantities sin P_m, sin Q_m, $r_{\beta m}$, and $r_{\alpha m}$, which may be either real or imaginary depending upon the value of c, occur only in the combinations $r_{\alpha m}^{\pm 1}$, sin P_m, and $r_{\beta m}^{\mp 1}$ sin Q. Since sin P_m is real or imaginary according as $r_{\alpha m}$ is real or imaginary, and sin Q_m is similarly related to $r_{\beta m}$, these combinations are always real for real values of c. With regard to the real or imaginary properties of its elements the matrices a_m then have the form

$$a_m = \begin{bmatrix} \mathrm{R} & \mathrm{I} & \mathrm{R} & \mathrm{I} \\ \mathrm{I} & \mathrm{R} & \mathrm{I} & \mathrm{R} \\ \mathrm{R} & \mathrm{I} & \mathrm{R} & \mathrm{I} \\ \mathrm{I} & \mathrm{R} & \mathrm{I} & \mathrm{R} \end{bmatrix}$$

where an R indicates a real quantity (not, of course, the same quantity in all positions) and an I indicates an imaginary quantity. The product of any two matrices of this form is also a matrix of the same form; hence of the elements of A occurring in equations (3.2):

A_{11}, A_{22}, A_{31}, and A_{42} are real;
A_{12}, A_{21}, A_{32}, and A_{41} are imaginary.

By definition a "surface" wave is one whose amplitude diminishes for large values of z, which means in our case that $r_{\alpha n}$ and $r_{\beta n}$ must be imaginary, that is, we are concerned only with values of $c < \beta_n$. Then, referring to equation (3.2), all terms of K and N are real and all terms of L and M are imaginary. Thus the ratio $\dot{u}_0/\dot{w}_0$ will always be imaginary, which means a phase difference of 90° between the horizontal and vertical displacements at the free surface. The particle motion is therefore an ellipse whose axes are vertical and horizontal. The phase difference may, however, be of either sign, and hence the sense of the motion around the ellipse is not necessarily retrograde with respect to the direction of propagation at all frequencies, as is the case with Rayleigh waves on a homogeneous medium.

If we were dealing with a dissipative medium, it would be necessary to consider complex values of c and k. In that case, the ratio $\dot{u}_0/\dot{w}_0$ would not necessarily be a pure imaginary, meaning that phase differences of other than $\pm 90°$ could occur and the axes of the displacement ellipse could be inclined from the vertical. It is therefore possible that imperfect elasticity of the medium is the cause of the inclination of the axes that is very commonly observed in the case of explosion excited surface waves on poorly consolidated sediments.

It is obvious that if two adjacent layers have identical physical properties, they must be equivalent to a single layer whose thickness is equal to the sum of the thicknesses of the two layers. Thus, if we let $a_m(d)$ be the matrix a_m computed for a given layer thickness d, we must have

$$a_m(d_1)a_m(d_2) = a_m(d_1 + d_2) \tag{3.3}$$

This relation may be readily verified by direct multiplication. Also, since k occurs in a_m only as the product kd_m, equation (3.3) implies,

$$a_m(k_1)a_m(k_2) = a_m(k_1 + k_2) \tag{3.4}$$

where k_1 and k_2 are any two values of k and a_m is computed for fixed values of c and d_m.

Asymptotic Form for Long Wave Lengths

As the wave length becomes very large, $kd_m \to 0$ and all the matrices a_m approach the unit matrix. Thus $J_n \to E_n^{-1}$ and equation (2.21) reduces to

$$\dot{u}_0/\dot{w}_0 = -(\gamma_n - 1)/\gamma_n r_{\alpha n} = \gamma_n r_{\beta n}/(\gamma_n - 1) \tag{4.1}$$

or

$$(\gamma_n - 1)^2 + \gamma_n^2 r_{\alpha n} r_{\beta n} = 0 \tag{4.2}$$

which is the equation for the Rayleigh wave velocity on the semi-infinite layer.

If we expand the terms of the matrices a_m in powers of k and ignore powers higher than the first, the matrix A reduces to

$$A \to \begin{bmatrix} 1 & ik\sum_1^{n-1} d_m & 0 & ik\sum_1^{n-1} d_m/\rho_m\beta_m^2 \\ ik\sum_1^{n-1} d_m[1-2(\beta_m/\alpha_m)^2] & 1 & ik\sum_1^{n-1} d_m/\rho_m\alpha_m^2 & 0 \\ 0 & ikc^2\sum d_m\rho_m & 1 & ik\sum_1^{n-1} d_m \\ ikc^2\sum_1^{n-1} d_m\rho_m[1-2\gamma_m+2\gamma_m(\beta_m/\alpha_m)^2] & 0 & ik\sum_1^{n-1} d_m[1-2(\beta_m/\alpha_m)^2] & 1 \end{bmatrix} \tag{4.3}$$

Thus for wave lengths so long that $k^2(\sum_1^{n-1} d_m)^2$ may be ignored, the a_m's commute and the order in which the upper layers are arranged is immaterial. To this order of approximation the quantities K, L, M, and N are linear functions of k. Hence for a given value of c equation (3.1) is a quadratic in k and may be solved explicitly.

Asymptotic Form for Short Wave Lengths

It has been shown by Sezawa and Kanai[3] that in the two-layer case the high-frequency asymptotic form of the phase velocity equation may be factored. One of these factors has a root corresponding to the Rayleigh wave velocity on the free surface of the first layer; the other is Stoneley's[4] expression for the velocity of interface waves on the contact between the two layers. The latter may or may not have a real root, depending upon the relative values of ρ, α, and β in the two layers. It seems evident on physical grounds that in the multilayer case the phase velocity equation must also be factorable at sufficiently high frequencies, the various factors representing Rayleigh waves on the free surface and Stoneley waves on each interface. In order to demonstrate this it will be convenient to write the matrix J in the form,

$$J = b_{n-1}\, b_{n-2} \cdot \cdot \cdot b_1 E_1^{-1} \tag{5.1}$$

where

$$b_m = E_{m+1}^{-1}\, D_m \tag{5.2}$$

That is, instead of grouping the matrix factors of J by layers, we now group them by interfaces. Now suppose that $c < \beta_{n-1}$, so that P_{n-1} and Q_{n-1} are imaginary and the sines and cosines represent hyperbolic functions. Then for large values of kd_{n-1}, $\sin P_{n-1} \rightarrow -i \cos P_{n-1}$ and $\sin Q_{n-1} \rightarrow -i \cos Q_{n-1}$. In this limit, the elements of b_{n-1} approach the following values:

$$\begin{aligned}
(b_{n-1})_{11} &= -(b_{n-1})_{12} = (\alpha_{n-1}/\alpha_n)^2\{\gamma_n - (\gamma_{n-1} - 1)(\rho_{n-1}/\rho_n)\} \cos P_{n-1}\\
(b_{n-1})_{13} &= -(b_{n-1})_{14} = (c/\alpha_n)^2\gamma_{n-1} r_{\beta(n-1)}\{\gamma_n - \gamma_{n-1}(\rho_{n-1}/\rho_n)\} \cos Q_{n-1}\\
(b_{n-1})_{21} &= -(b_{n-1})_{22} = (\alpha_{n-1}/\alpha_n)^2(r_{\alpha(n-1)}/r_{\alpha n})\{(\gamma_n - 1) - \gamma_{n-1}(\rho_{n-1}/\rho_n)\} \cos P_{n-1}\\
(b_{n-1})_{23} &= -(b_{n-1})_{24} = -(c/\alpha_n)^2(\gamma_{n-1}/r_{\alpha n})\{(\gamma_n - 1) - (\gamma_{n-1} - 1)(\rho_{n-1}/\rho_n)\} \cos Q_{n-1}\\
(b_{n-1})_{31} &= -(b_{n-1})_{32} = -(\alpha_{n-1}/c)^2(\gamma_n r_{\beta n})^{-1}\{(\gamma_n - 1) - (\gamma_{n-1} - 1)(\rho_{n-1}/\rho_n)\} \cos P_{n-1}\\
(b_{n-1})_{33} &= -(b_{n-1})_{34} = (\gamma_{n-1} r_{\beta(n-1)}/\gamma_n r_{\beta n})\{(\gamma_n - 1) - \gamma_{n-1}(\rho_{n-1}/\rho_n)\} \cos Q_{n-1}\\
(b_{n-1})_{41} &= -(b_{n-1})_{42} = (\alpha_{n-1}/c)^2(r\alpha_{(n-1)}/\gamma_n)\{\gamma_n - \gamma_{n-1}(\rho_{n-1}/\rho_n)\} \cos P_{n-1}\\
(b_{n-1})_{43} &= -(b_{n-1})_{44} = (\gamma_{n-1}/\gamma_n)\{\gamma_n - (\gamma_{n-1} - 1)(\rho_{n-1}/\rho_n)\} \cos Q_{n-1}
\end{aligned}$$

If we set $J_{n-1} = b_{n-2}\, b_{n-3} \cdot \cdot \cdot b_1 E_1^{-1}$, then since $(b_{n-1})_{j1} = -(b_{n-1})_{j2}$ and $(b_{n-1})_{j3} = -(b_{n-1})_{j4}$ for high frequencies, we may write

$$\begin{aligned}
J_{22} - J_{12} &= [(b_{n-1})_{11} - (b_{n-1})_{21}][(J_{n-1})_{22} - (J_{n-1})_{12}]\\
&\qquad + [(b_{n-1})_{13} - (b_{n-1})_{23}]\,[(J_{n-1})_{42} - (J_{n-1})_{32}]\\
J_{11} - J_{21} &= [(b_{n-1})_{11} - (b_{n-1})_{21}][(J_{n-1})_{11} - (J_{n-1})_{21}]\\
&\qquad + [(b_{n-1})_{13} - (b_{n-1})_{23}]\,[(J_{n-1})_{31} - (J_{n-1})_{41}]\\
J_{42} - J_{32} &= [(b_{n-1})_{31} - (b_{n-1})_{41}][(J_{n-1})_{22} - (J_{n-1})_{12}]\\
&\qquad + [(b_{n-1})_{33} - (b_{n-1})_{43}]\,[(J_{n-1})_{42} - (J_{n-1})_{32}]\\
J_{31} - J_{41} &= [(b_{n-1})_{31} - (b_{n-1})_{41}][(J_{n-1})_{11} - (J_{n-1})_{21}]\\
&\qquad + [(b_{n-1})_{33} - (b_{n-1})_{43}]\,[(J_{n-1})_{31} - (J_{n-1})_{41}]
\end{aligned}$$

[3] K. Sezawa and K. Kanai, "Anomalous Dispersion of Elastic Surface Waves," *Bull. Earthq. Res. Inst. Tokyo*, 16: 683 (1938).

[4] R. S. Stoneley, "Elastic Waves at the Surface of Separation of Two Solids," *Proc. Roy. Soc.*, 106: 416 (1924).

Setting

$$\left.\begin{aligned} K' &= (J_{n-1})_{22} - (J_{n-1})_{12} \\ L' &= (J_{n-1})_{11} - (J_{n-1})_{21} \\ M' &= (J_{n-1})_{42} - (J_{n-1})_{32} \\ N' &= (J_{n-1})_{31} - (J_{n-1})_{41} \end{aligned}\right\} \tag{5.3}$$

$$\left.\begin{aligned} R &= (b_{n-1})_{11} - (b_{n-1})_{21} \\ S &= (b_{n-1})_{13} - (b_{n-1})_{23} \\ T &= (b_{n-1})_{31} - (b_{n-1})_{41} \\ U &= (b_{n-1})_{33} - (b_{n-1})_{43} \end{aligned}\right\} \tag{5.4}$$

the relation (2.21) between the elements of J may be written in the form

$$\frac{RK' + SM'}{RL' + SN'} = \frac{TK' + UM'}{TL' + UN'}$$

Cross-multiplication and cancellation reduces this equation to the factored form,

$$(RU - ST)\,(K'N' - L'M') = 0 \tag{5.5}$$

Equating the first factor of this expression to zero and using the values of the elements of b_{n-1} given above, the common factor $\cos P_{n-1} \cos Q_{n-1}$ may be divided out and the resulting expression put in the form,

$$\begin{aligned} \rho_n^2[(\gamma_n - 1)^2 + \gamma_n^2 r_{\alpha n} r_{\beta n}][1 + r_{\alpha(n-1)} r_{\beta(n-1)}] - 2\rho_n\rho_{n-1}[(\gamma_{n-1}) + \gamma_n r_{\alpha n} r_{\beta n}][(\gamma_{n-1} - 1) \\ + \gamma_{n-1} r_{\alpha(n-1)}\, r_{\beta(n-1)}] + \rho_n\rho_{n-1}[r_{\alpha(n-1)} r_{\beta n} + r_{\alpha n} r_{\beta(n-1)}] + \rho_{n-1}^2[(\gamma_{n-1} - 1)^2 \\ + \gamma_{n-1}^2 r_{\alpha(n-1)} r_{\beta(n-1)}][1 + r_{\alpha n} r_{\beta n}] = 0 \end{aligned} \tag{5.6}$$

which is equivalent to Stoneley's equation for the $(n - 1)^{\text{th}}$ interface.

The second factor of equation (5.5), when equated to zero, is equivalent to the original expression, except that it refers to $(n - 1)$ instead of n layers. The same process may therefore be repeated, leading to a product of factors each corresponding to Stoneley waves on one of the interfaces, and a final factor which has the form of equation (4.2) for the first layer and thus represents Rayleigh waves on the free surface of this layer.

It will be noted that a complete algebraic factorization at high frequencies by the foregoing process can be carried through only when c is less than the smallest value of β_m. For values of c greater than the minimum value of β_m there will be at least one layer for which Q_m is a real quantity. In this case, the ratios K/L and M/N will not apporach an asymptotic value for large values of k, but will remain oscillatory functions of k. Thus for a given value of c greater than the smallest value of β_m and less than β_n there will generally be an infinite number of values of k for which $KN - ML = 0$. These roots represent the sequence of normal-mode solutions. Since there is only one root at very low frequency, each of these higher order modes must have a low frequency cut-off.

THE MATRIX a_m FOR A FLUID LAYER

In some cases of seismological interest one of the upper layers may be a fluid. If we go directly to the limit $\beta_m = 0$, a difficulty arises because the matrix E_m as defined by equation (2.12) becomes singular and E_m^{-1} does not exist. However, we may define an *effective* inverse transformation as follows:

Setting

$$\begin{aligned} \beta_m &= 0, \text{equation (2.11) becomes} \\ \dot{u}_{m-1}/c &= -(\alpha_m/c)^2(\Delta_m' + \Delta_m'') \\ \dot{w}_{m-1}/c &= -(\alpha_m/c)^2 r_{\alpha m}(\Delta_m' - \Delta_m'') \\ \sigma_{m-1} &= \rho_m \alpha_m^2(\Delta_m' + \Delta_m'') \\ \tau_{m-1} &= 0 \end{aligned}$$

Since in an ideal fluid continuity of the tangential displacement at a solid boundary is not a required condition, the first of these equations is irrelevant and we may write,

$$\Delta_m' + \Delta_m'' = \sigma_{m-1}/\rho_m\alpha_m^2$$

$$\Delta_m' - \Delta_m'' = -(c/\alpha_m)^2 r_{\alpha m}^{-1}\, \dot{w}_{m-1}/c$$

Also, since rotational waves do not exist in the fluid, $\omega_m' = \omega_m'' = 0$. The transformation $(\Delta_m' + \Delta_m'', \Delta_m' - \Delta_m'', \omega_m' - \omega_m'', \omega_m' + \omega_m'') \rightarrow (\dot{u}_{m-1}/c, \dot{w}_{m-1}/c, \sigma_{m-1}, \tau_{m-1})$, which is the effective inverse of E_m, therefore has the matrix,

$$F_m^{-1} = \begin{bmatrix} 0 & 0 & (\rho_m\alpha_m^2)^{-1} & 0 \\ 0 & -(c/\alpha_m)^2 r_{\alpha m}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.1}$$

For $\beta_m = 0$ the matrix D_m takes the form

$$D_m = \begin{bmatrix} (-\alpha_m/c)^2 \cos P_m & i(\alpha_m/c)^2 \sin P_m & 0 & 0 \\ i(\alpha_m/c)^2 r_{\alpha m} \sin P_m & -(\alpha_m/c)^2 r_{\alpha m} \cos P_m & 0 & 0 \\ \rho_m\alpha_m^2 \cos P_m & -i\rho_m\alpha_m^2 \sin P_m & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.2}$$

from which we find

$$a_m = D_m F_m^{-1} = \begin{bmatrix} 0 & i r_{\alpha m}^{-1} \sin P_m & -(\rho_m c^2)^{-1} \cos P_m & 0 \\ 0 & \cos P_m & i r_{\alpha m} (\rho_m c^2)^{-1} \sin P_m & 0 \\ 0 & i \rho c^2 r_{\alpha m}^{-1} \sin P_m & \cos P_m & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{6.3}$$

Computational Procedure

Numerical computation of the function $c(k)$ from equation (3.1) must be carried out by a process of trial and error. It will usually be convenient to take the thickness of the first layer as the unit of length, ρ_1 as the unit of density, and β_1 (or α_1 if the first layer is a fluid) as the unit of velocity, in which case the result of the computation will be a relation between the dimensionless quantities c/β_1 and kd_1. Since the coefficients of the functions of P_m and Q_m in the matrices a_m depend only on c and the constants of the medium, and are independent of k, less computation is required to determine the value of k for a given value of c than for the reverse process. The limits of possible values of c are given by the Rayleigh velocities for the n^{th} layer and for the lowest velocity layer present. A set of values of c at convenient intervals between these limits may then be chosen and the quantities $r_{\alpha m}$, $r_{\beta m}$, and γ_m for each layer computed for each value of c.

From previously computed $c(k)$ functions for two-layer cases (e.g., Sezawa's curves) it will usually be possible to make a preliminary estimate of the value of k for a given value of c that will be at least within an order of magnitude of the correct value. With this as a first trial value, the elements of the matrices a_m, the pertinent elements of the product matrix A, and the ratios K/L and M/N may be computed. By repeating the computation with other assumed values of k and plotting the values $K/L - M/N$ against k it will usually be possible to determine the root to one-tenth of one per cent with four or five trial values. Since the computations involve taking differences between quantities of comparable magnitude at several stages, it is advisable to carry at least two more significant figures in the computation than is desired in the final value of k.

Alternative Formulas for the Two-Layer Case

If the expressions (3.2) are substituted in the equation $KN - LM = 0$, certain terms in the products cancel out and the equation may be written in the form,

$$\begin{aligned}&[(\gamma_n - 1)^2 + \gamma_n^2 r_{\alpha n} r_{\beta n}][A_{21}A_{12} - A_{11}A_{22}] + (\rho_n c^2)^{-1}[(\gamma_n - 1) + \gamma_n r_{\alpha n} r_{\beta n}] \\ &\quad [A_{12}A_{41} + A_{22}A_{31} - A_{11}A_{42} - A_{21}A_{32}] + r_{\alpha n}(\rho_n c^2)^{-1}[A_{12}A_{31} - A_{11}A_{32}] \\ &\quad + r_{\beta n}(\rho_n c^2)^{-1}[A_{21}A_{42} - A_{22}A_{41}] + (\rho_n c^2)^{-2}(1 + r_{\alpha n} r_{\beta n})[A_{31}A_{42} - A_{41}A_{32}] = 0\end{aligned} \tag{8.1}$$

In the case of more than two layers there does not appear to be any advantage for numerical computations in writing the phase velocity equation in this form, but in the two-layer case ($n = 2$, $A = a_1$) insertion of the explicit expressions for the elements of the matrix a leads to the more convenient form,

$$C_1 - C_2 \cos P_1 \cos Q_1 + C_3 r_{\alpha 1}^{-1} r_{\beta 1}^{-1} \sin P_1 \sin Q_1 - C_4 r_{\beta 1}^{-1} \cos P_1 \sin Q_1$$
$$- C_5 r_{\alpha 1}^{-1} \sin P_1 \cos Q_1 = 0 \tag{8.2}$$

where the coefficients C_1 through C_5 are given by the following expressions:

$$\begin{aligned}
C_1 &= 2\gamma_1(\gamma_1 - 1)(A - D + E) \\
C_2 &= A + C_1 \\
C_3 &= A([\gamma_1 - 1]^2 + \gamma_1^2 r_{\alpha 1}^2 r_{\beta 1}^2) - 2B(\rho_1/\rho_2)([\gamma_1 - 1]^3 + \gamma_1^3 r_{\alpha 1}^2 r_{\beta 1}^2) \\
&\quad + (\rho_1/\rho_2)^2(1 + r_{\alpha 2} r_{\beta 2})([\gamma_1 - 1]^4 + \gamma_1^4 r_{\alpha 1}^2 r_{\beta 1}^2) \\
C_4 &= i(\rho_1/\rho_2)([\gamma_1 - 1]^2 r_{\beta 2} + \gamma_1^2 r_{\alpha 2} r_{\beta 1}^2) \\
C_5 &= i(\rho_1/\rho_2)([\gamma_1 - 1]^2 r_{\alpha 2} + \gamma_1^2 r_{\alpha 1}^2 r_{\beta 2}) \\
A &= (\gamma_2 - 1)^2 + \gamma_{\alpha 2} r_{\alpha 2} r_{\beta 2} \\
B &= (\gamma_2 - 1) + \gamma_2 r_{\alpha 2} r_{\beta 2} \\
D &= (\rho_1/\rho_2)(2\gamma_1 - 1)\, B \\
E &= (\rho_1/\rho_2)^2(1 + r_{\alpha 2} r_{\beta 2})\gamma_1(\gamma_1 - 1)
\end{aligned}$$

Equation (8.2) is particularly useful because it can be solved explicitly for the higher-order roots. For $\beta_1 < c < \alpha_1$, P_1 is imaginary and Q_1 is real, so that $\sin P_1$ and $\cos P_1$ become very large in absolute value for moderately large values of kd_1, while $|\sin Q_1|$ and $|\cos Q_1|$ remain $\leqq 1$. Dividing equation (8.2) through by $\cos P_1 \cos Q_1$, dropping the small term $C_1/\cos P_1$, and noting that $\tan P_1 \rightarrow -i$ for large values of kd_1, we find,

$$\tan Q_1 \rightarrow r_{\beta 1}(i r_{\alpha 1}^{-1} C_5 - C_2)/(i r_{\alpha 1}^{-1} C_3 + C_4) \tag{8.3}$$

or

$$kd_1 \rightarrow r_{\beta 1}^{-1} \tan^{-1}\{r_{\beta 1}(i r_{\alpha 1}^{-1} C_5 - C_2)/(i r_{\alpha 1}^{-1} C_3 + C_4)\} + n\pi r_{\beta 1}^{-1} \tag{8.4}$$

where $n = 1, 2, 3, \cdots$ etc. and $\tan^{-1}$ is given its smallest positive value.

Love Waves

In the case of Love waves the boundary conditions to be satisfied at each interface are continuity of the transverse component of displacement, v, and of the transverse shear stress, Y_z. The pertinent plane-wave solution of the elastic equations of motion for a homogeneous layer is

$$u = w = 0$$

$$v = \exp[i(pt - kx)][v' \exp(-ikr_\beta z) + v'' \exp(ikr_\beta z)] \tag{9.1}$$

where v' and v'' are constants.

The corresponding transverse shearing stress is

$$Y_z = \mu \partial v/\partial z = ik\mu r_\beta \exp\,[i(pt - kx)]\,[-v' \exp\,(-ikr_\beta z) + v'' \exp\,(ikr_\beta z)] \tag{9.2}$$

At the $(m-1)^{\text{th}}$ interface we then have

$$\left.\begin{aligned} (\dot{v}/c)_{m-1} &= ik(v_m' + v_m'') \\ (Y_z)_{m-1} &= ik\mu_m r_{\beta m}(v_m'' - v_m') \end{aligned}\right\} \tag{9.3}$$

and at the m^{th} interface

$$\left.\begin{aligned} (\dot{v}/c)_m &= (v_m' + v_m'')ik \cos Q_m - (v_m'' - v_m')k \sin Q_m \\ (Y_z)_m &= -(v_m' + v_m'')k\mu r_{\beta m} \sin Q_m + (v_m'' - v_m')ik\mu_m r_{\beta m} \cos Q_m \end{aligned}\right\} \tag{9.4}$$

By eliminating v_m' and v_m'' between equations (9.3) and (9.4),

$$\left.\begin{aligned} (\dot{v}_m/c)_m &= (\dot{v}/c)_{m-1} \cos Q_m + (Y_z)_{m-1}\mu_m^{-1} r_{\beta m}^{-1} i \sin Q_m \\ (Y_z)_m &= (\dot{v}/c)_{m-1} i\mu_m r_{\beta m} \sin Q_m + (Y_z)_{m-1} \cos Q_m \end{aligned}\right\} \tag{9.5}$$

The matrix a_m in this case is therefore

$$\begin{aligned} a_m &= \begin{bmatrix} \cos Q_m & i\mu_m^{-1} r_{\beta m}^{-1} \sin Q_m \\ i\mu_m r_{\beta m} \sin Q_m & \cos Q_m \end{bmatrix} \\ &= \cos Q_m \begin{bmatrix} 1 & i\mu_m^{-1} r^{-1}{}_{\beta m} \tan Q_m \\ i\mu_m r_{\beta m} \tan Q_m & 1 \end{bmatrix} \end{aligned} \tag{9.6}$$

Setting $a_{n-1}\, a_{n-2} \cdot\cdot\cdot a_1 = A$ as before, the analog of equations (2.18) is

$$\left.\begin{aligned} (\dot{v}/c)_{n-1} &= A_{11}(\dot{v}/c)_0 + A_{12}(Y_z)_0 \\ (Y_z)_{n-1} &= A_{21}(\dot{v}/c)_0 + A_{22}(Y_z)_0 \end{aligned}\right\} \tag{9.7}$$

and using equation (9.3) for $m = n$

$$\begin{aligned} v_n' + v_n'' &= A_{11}(ik)^{-1}(\dot{v}/c)_0 + A_{12}(ik)^{-1}(Y_z)_0 \\ v_n'' - v_n' &= A_{12}(ik\mu_n r_{\beta n})^{-1}(\dot{v}/c)_0 + A_{22}(ik\mu_n r_{\beta n})^{-1}(Y_z)_0 \end{aligned} \tag{9.8}$$

The conditions for the existence of free surface waves are $(Y_z)_0 = 0$ and $v_n'' = 0$, which, with equations (9.8) lead to

$$A_{21} = -\mu_n r_{\beta n} A_{11} \tag{9.9}$$

In the two-layer case, $A = a_1$, and equation (9.9) reduces to the Love wave dispersion equation in the familiar form.

$$\tan Q_1 = -i(\mu_2 r_{\beta 2}/\mu_1 r_{\beta 1}) \tag{9.10}$$

Application to the Rayleigh Waves of Earthquakes in Continental Areas

Considerably more detailed study of the dispersion of Rayleigh waves over continental paths will be required before it will be possible to make any conclusive quantitative interpretation of the observed dispersion in terms of continental structure. However, as a preliminary step in this direction, phase and group velocities have been computed for assumed layered structures as shown in table 1. The values of α, β, and d have been so chosen that the time-distance curves for the first arrivals of P and S waves are the same in all cases except for that portion of the curve that corresponds to refraction from the intermediate layer. Case III is intended to be typical of the interpretations that have been made from blast and near earthquake recordings in areas of little or no low velocity sedimentary cover, in particular those

TABLE 1

	Layer	α (km/sec.)	β(km/sec.)	ρ(gm/cm³)	d (km.)
Case I....................	1	6.14	3.39	2.70	13.60
	2	5.50	3.18	2.70	11.85
	3	8.26	4.65	3.00	∞
Case II....................	1	6.14	3.39	2.70	28.38
	2	8.26	4.65	3.00	∞
Case III..................	1	6.14	3.39	2.70	13.60
	2	7.00	4.04	2.70	21.21
	3	8.26	4.65	3.00	∞

TABLE 2

	Layer	α	β	ρ	d
Case I.................	1	1.810	1.000	1.000	1.000
	2	1.620	0.938	1.000	0.871
	3	2.440	1.370	1.110	∞
Case II................	1	1.810	1.000	1.000	1.000
	2	2.440	1.370	1.110	∞
Case III...............	1	1.810	1.000	1.000	1.000
	2	2.060	1.190	1.000	1.560
	3	2.440	1.370	1.110	∞

TABLE 3

Case I

c/β	kd_1	$k(d_1 + d_2)$	$\dot{u}_0/\dot{w}_0$
1.265	0	0	−0.670 i
1.250	0.0647	0.1211	−0.744 i
1.200	0.405	0.758	−0.849 i
1.150	0.729	1.364	−0.716 i
1.100	0.962	1.800	−0.645 i
1.050	1.186	2.219	−0.619 i
1.000	1.461	2.734	−0.619 i
0.950	1.920	3.592	−0.638 i
0.938	2.126	3.978	−0.645 i
0.920	2.817	5.271	−0.662 i
0.920	5.00	9.36	−0.669 i
0.924	∞	∞	−0.667 i

Case II

Rayleigh mode		First mode	
c/β_1	kd_1	c/β_1	kd_1
1.265	0	1.370	2.383
1.250	0.1339	1.350	2.957
1.200	0.895	1.300	4.303
1.150	1.550	1.250	5.510
1.100	2.029	1.150	7.716
1.050	2.518	1.100	9.397
1.000	3.167	1.050	12.766
0.950	4.489	1.030	15.882
0.930	6.381	1.010	25.54
0.924	∞	1.000	∞

Case III

c/β_1	kd_1	$k(d_1 + d_2)$	$\dot{u}_0/\dot{w}_0$
1.265	0	0	−0.670 i
1.250	0.0717	0.1836	−0.710 i
1.200	0.5123	1.311	−0.828 i
1.150	0.8634	2.210	−0.746 i
1.100	1.197	3.064	−0.699 i
1.050	1.551	3.971	−0.672 i
1.025	1.954	5.002	−0.656 i
1.000	2.365	6.054	−0.649 i
0.975	2.929	7.498	−0.647 i
0.950	3.837	9.823	−0.652 i
0.930	5.76	14.75	−0.662 i
0.924	∞	∞	−0.667 i

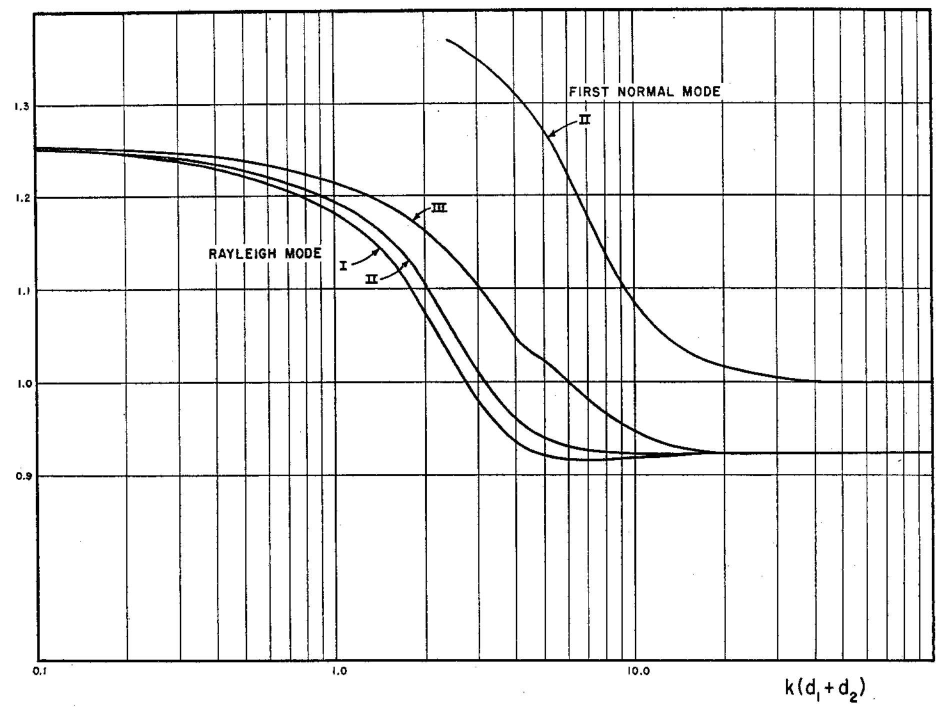

Fig. 2. Phase velocity of Rayleigh waves for assumed crustal structures.

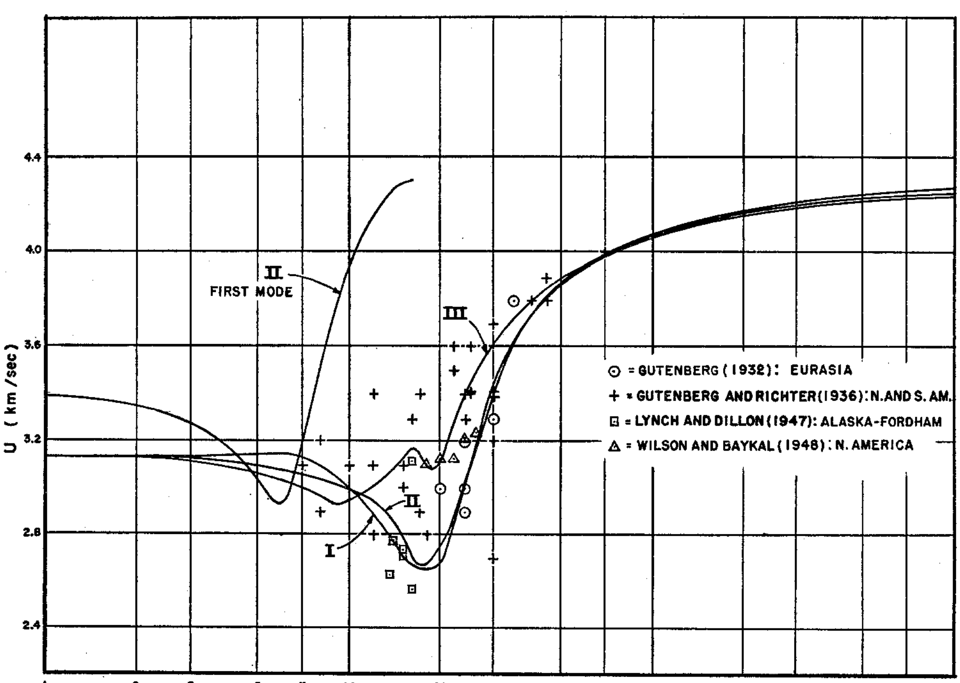

Fig. 3. Group velocity of Rayleigh waves for assumed crustal structures.

of Leet, Hodgson, and Tuve.[5] Case I has been computed to show the effect on Rayleigh wave dispersion of a low velocity zone whose existence has been suggested by Gutenberg,[6] and II represents an intermediate case between I and III for comparison purposes. The computations have been carried through in dimensionless form with β_1, ρ_1, and d taken respectively as the units of velocity, density, and length in each case. The values of the ratios actually used are given in table 2.

The computed values of kd_1, $k(d_1 + d_2)$, and of $\dot{u}_0/\dot{w}_0$ for various values of c/β_1 are listed in table 3. With the sign conventions used here a negative imaginary value of $\dot{u}_0/\dot{w}_0$ corresponds to retrograde particle motion. Only the mode of lowest order (Rayleigh mode) has been computed for Cases I and III, but the next higher mode is also given for Case II.

The phase velocities of the Rayleigh modes for the three cases and the first normal mode for Case II are plotted in dimensionless units in figure 2. The curve for Case I has a minimum at $c/\beta_1 = .918$, $k(d_1 + d_2) = 7$ approximately.

The corresponding group velocities, computed from the expression $U = c + kdc/dk$ by graphical differentiation, are plotted against the period $T = 2\pi/kc$ in figure 3. Although the computation of the roots corresponding to the first normal mode has not been carried through for Cases I and III, rough estimates indicate that the low frequency cut-off for these cases will not be very different from that for Case II.

Some observed values of group velocities at various frequencies over continental paths are also plotted in figure 3. These values have been taken from tabulations published by Gutenberg, Gutenberg and Richter, Lynch and Dillon, and Wilson and Baykal.[7] In view of the scatter of the observed points it is not possible to say that any one of the assumed models is conspicuously favored over the others, but the data do not disprove the hypothesis of the possible existence of a low-velocity layer under at least some parts of the continents. Some of the scatter of the observed values is no doubt due to observational errors such as misidentification of wave type and erroneous determination of periods due to interference by the simultaneous arrival of higher frequencies in the higher-order modes, but some of the scatter is probably due to real horizontal inhomogeneity of the continental crusts. Certainly the discordance between the crustal structures derived from P and S travel-time data from blasts and near earthquakes in different areas suggests a comparatively high degree of heterogeneity in the layers above the Mohorovičić discontinuity.

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