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SYNTHESIS OF DISPERSED SURFACE WAVES BY MEANS OF FOURIER TRANSFORM

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ABSTRACT

Flexural disturbances propagated upon a thin aluminum plate were studied experimentally. The thickness of the plate is 1.0 mm., and the distances between the source and the observation points are $r = 0$, 3.0, 5.0, 7.0, 15.0, and 20.0 cm.

Using the method of Fourier transform, the dispersion curve was obtained. It agrees well with the theory, and the thickness of the plate was estimated with good accuracy (0.981 mm.).

Disturbances at the distances 7.0, 3.0, and 0 cm. were numerically reproduced by means of Fourier synthesis from the data obtained at $r = 15$ cm. and 20 cm., and were compared with the observed disturbances. The agreement of the two kinds of curves are fairly good, with a little larger discrepancy for the case $r = 0$ than for the others.

Introduction

First, consider a problem of the propagation of impulsive disturbance given at a point in some dispersive medium. According to Fourier's integral theorem, any disturbance which is aperiodic with regard to time and is subject to certain conditions is expressed by the superposition of component waves with respective frequencies. Since each component wave has its own propagation velocity, the disturbance is dispersed and the wave form changes during propagation. If, however, we know the dispersion formula and the mechanism of the origin, we can calculate this modified form of disturbance at any point.

This procedure is possible not only when we know the mechanism of the origin; given the law of propagation and the spectrum of the shock, we can deduce the disturbance at any point at any time—consequently the motion at the origin, too. This poses an interesting problem in the field of wave-propagation study, and the following is an example of investigation with respect to it.

Fundamental Formula

Formulas used for the calculation were prepared earlier,[1] and therefore only the necessary ones are given in the following, which were a little modified to cover more general ground than before.

The disturbance given at the origin is $f(t)$, which can be expressed as follows:

$$f(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f^*(p) \exp(ipt)\, dp \tag{2.1}$$

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[1] Y. Satô, "Analysis of Dispersed Surface Waves by Means of Fourier Transform," I, *Bull. Earthq. Res. Inst.*, 33: 33–48 (1955); II, *ibid.*, 34: 9–18 (1956); III, *ibid.*, 34: 131–138 (1956).

At the point r the disturbance is

$$f(t; r) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f^*(p; r) \exp(ipt)\, dp \tag{2.2}$$

which can be also expressed in the following way:

$$f(t; r) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} A(r_0, r; p) f^*(p; r_0) \exp\left\{ip\left(t - \frac{r - r_0}{V(p)}\right)\right\} dp \tag{2.3}$$

The factor $\exp\{-ip(r - r_0)/V(p)\}$ is the phase shift between the points r_0 and r (r_0 is the distance at which the surface wave is originated and is assumed to be very small); $f^*(p; r_0)$ is the Fourier transform of the movement at $r = r_0$ and is expressed as

$$f^*(p; r_0) = F(p; r_0) \exp\{-i\beta(p)\} \tag{2.4}$$

where $F(f; r_0)$ is assumed to be real and $-\beta(p)$ is the argument of $f^*(p; r_0)$.

$A(r_0, r; p)$ is the factor expressing the decrease of amplitude and the phase shift, and can be separated into the factors such as

$$A(r_0, r; p) = \exp\left\{-\frac{\pi}{QVT}(r - r_0)\right\} \bar{A}(r_0, r) \tag{2.5}$$

The first exponential factor gives the effect of the absorption of the medium. T is the period and Q is the number expressing the absorption characteristic. $\bar{A}(r_0; r)$ is the effect of the geometrical spread of the wave front, and therefore this function naturally satisfied the condition

$$\bar{A}(r_0, r) \cdot \bar{A}(r, r_A) = \bar{A}(r_0, r_A) \tag{2.6}$$

Combining the foregoing relations, we can get the following equation, which gives the movement at any point:

$$\begin{aligned} f(t; r) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} & F(p; r_A) \cdot \bar{A}(r_A, r_0) \exp\left\{\frac{p}{2QV}(r_A - r_0)\right\} \\ & \cdot \bar{A}(r_0, r) \exp\left\{\frac{p}{2QV}(r_0 - r)\right\} \\ & \cdot \exp\left\{-i\beta(p)\right\} \exp\left\{ip\left(t - \frac{r - r_0}{V(p)}\right)\right\} dp \end{aligned} \tag{2.7}$$

The first line of the integrand gives the spectrum at $r = r_0$. If the second line is multiplied with the first the result gives the spectrum at the point r. The functions

$\bar{A}(r_A, r_0)$ and $\bar{A}(r_0, r)$ are assumed to be independent from the frequency, and therefore they have nothing to do with the shape of the curve and are omitted in the following calculation.

The phase velocity V which is involved in the foregoing expression is obtained from the following formula

$$V(p) = p(r_A - r_B)/[-\arg f^*(p; r_A) + \arg f^*(p; r_B)] \tag{2.8}$$

and $\beta(p)$ from the equation

$$\arg f^*(p; r_A) = -\beta(p) - p(r_A - r_0)/V(p) \tag{2.9}$$

If the wave propagates one-dimensionally, or the wave front spreads slowly, the formula given above will suffice. However, if we discuss the wave form near the origin in the two-dimensional problem, we must take the effect of phase lag caused by the spread of wave front into account.

To compensate this effect we assume another factor

$$\exp\{-iL(r_0, r)\} \tag{2.10}$$

in the right-hand member of (2.5), and then we have from (2.7)

$$f(t; r) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(p; r_A) \cdot \bar{A}(r_A, r_0) \exp\left\{\frac{p}{2QV}(r_A - r_0)\right\} \cdot \bar{A}(r_0, r) \exp\left\{\frac{p}{2QV}(r_0 - r)\right\} \cdot \exp\{-i[\beta(p) + L(r_0, r)]\} \exp\left\{ip\left(t - \frac{r - r_0}{V(p)}\right)\right\} dp \tag{2.7'}$$

and both (2.8) and (2.9) suffer respective change.

If the plate is thin and the source is a simple shock applied in or upon the plate, the component wave will be proportional to the Bessel function J_0, and the phase difference between this and the cos-function is approximately $\pi/4$. This can be seen in the asymptotic expansion of J_0 function, but more easily by figure 1.

Data and Numerical Calculation

A similar investigation was carried out several years ago by the author,[2] using the data given by F. Kishinouye.[3] For the present study the data were obtained by means of the experiment of the propagation of flexural waves upon a thin aluminum sheet.

[2] Y. Satô, as cited in note 1, second paper.
[3] F. Kishinouye, "Studies on Lake Ice," *Bull. Earthq. Res. Inst.*, 21: 298–306 (1943).

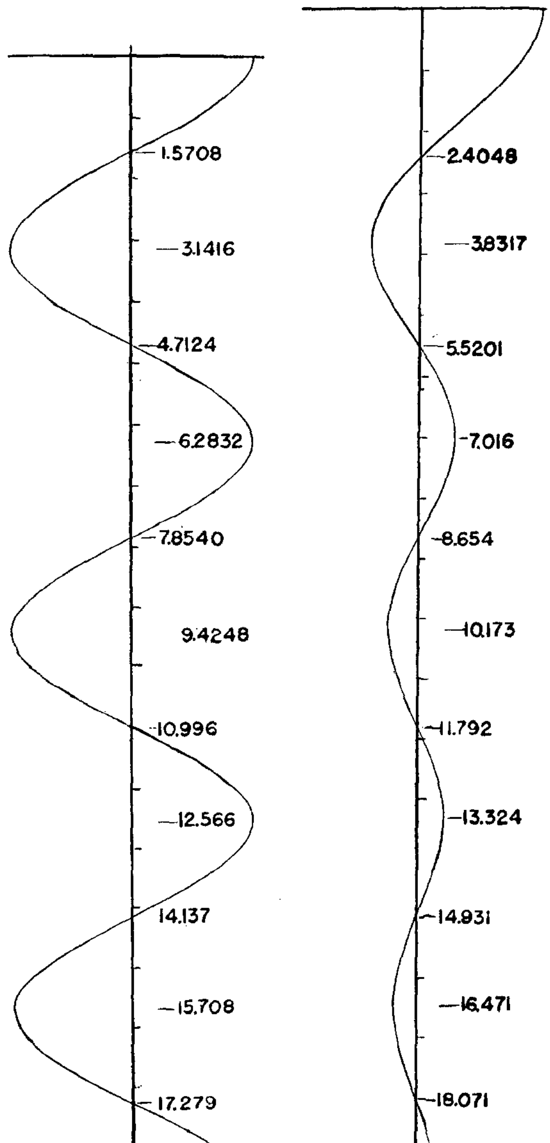

Fig. 1. Maxima, minima, and zero points of cos- (left) and J_0- (right) functions. Cos-function is shifted by $\pi/4$ so that the two curves may be better compared.

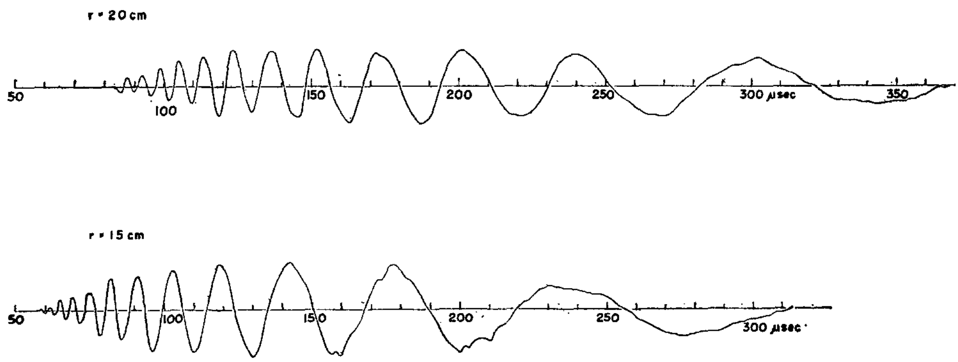

Fig. 2. Observed disturbances at the points $r = 15$ and 20 cm.

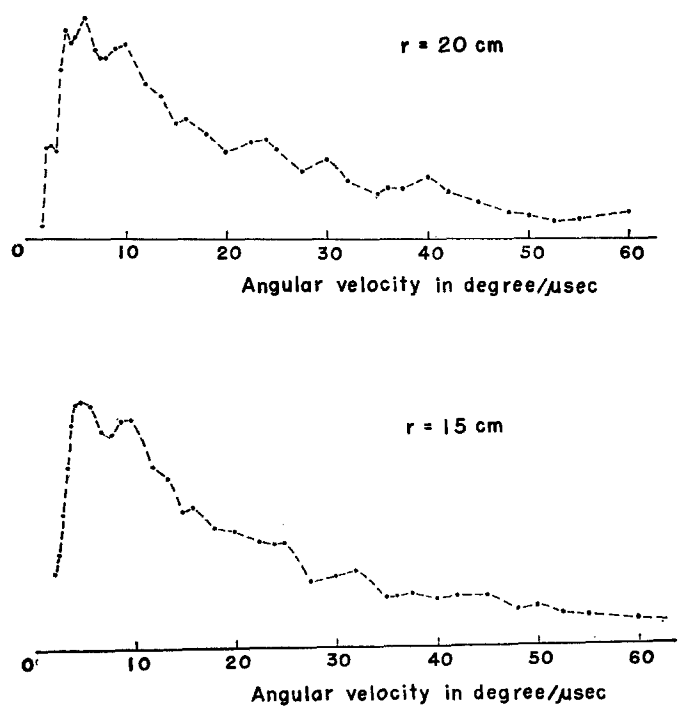

Fig. 3. Spectra of the observed curves given in figure 2.

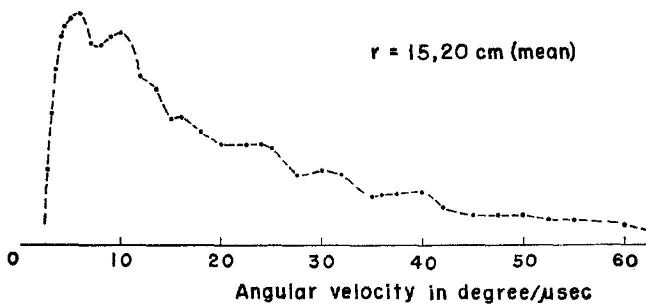

Fig. 4. Mean value of the two curves given in figure 3. This curve was used for the following numerical computation.

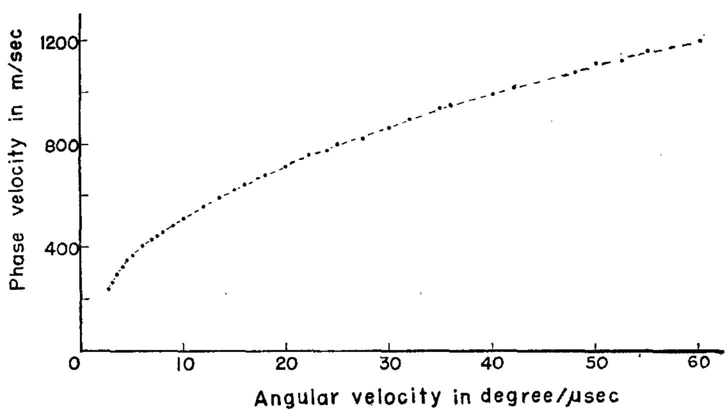

Fig. 5. Dispersion curve obtained by the application of the method of Fourier transform to the observation given in figure 2. (Linear scale.)

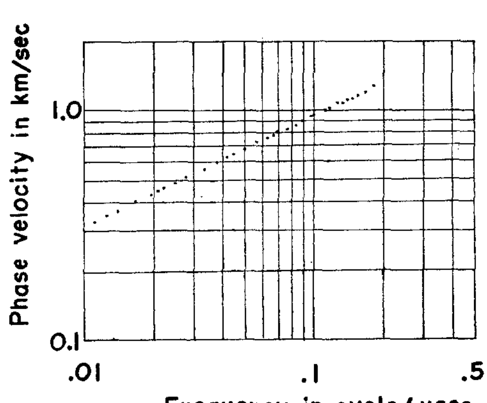

Fig. 6. Dispersion curve obtained by the application of the method of Fourier transform to the observation given in figure 2. (Log-log scale.)

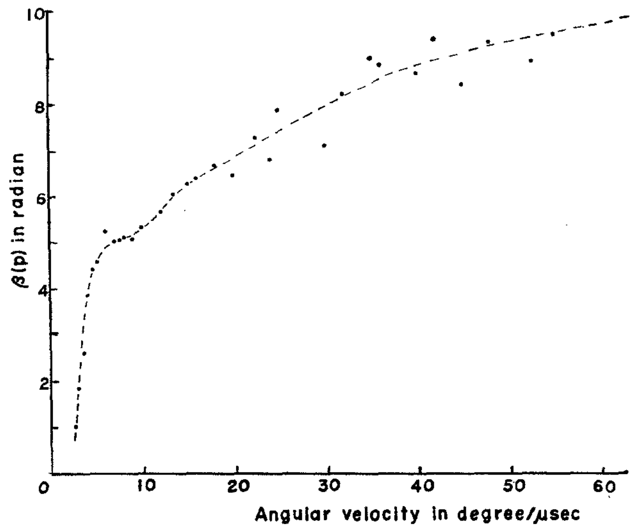

Fig. 7. Phase angle $\beta(p)$. Cf. formulas (2.4) and (2.9).

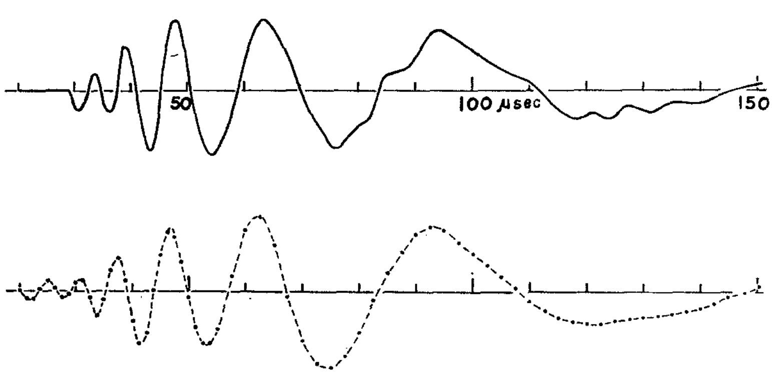

Fig. 8. Observed disturbance at the point $r = 7$ cm. (above) and the numerically reproduced curve (below), using the formula (2.7), which was based on the idea of Fourier transform.

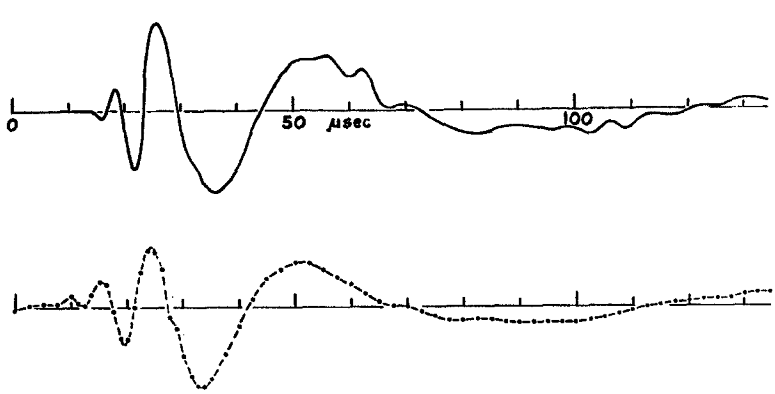

Fig. 9. Observed disturbance at the point $r = 3$ cm. (above) and the numerically reproduced curve (below).

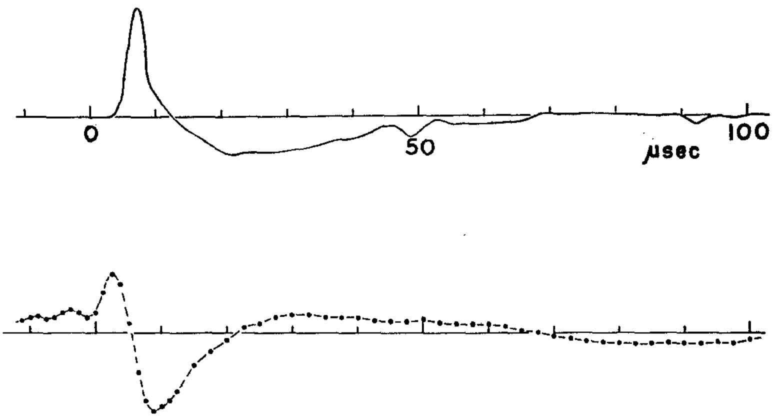

Fig. 10. Observed disturbance at the point $r = 0$ cm. (above). Pick-up was put from the rear side of the aluminum sheet at the opposite point of the transducer. The curve below ($r = 0$ cm., $L = 0$) was numerically reproduced, using the same simple formula (2.7), as for the two curves in figures 8 and 9.

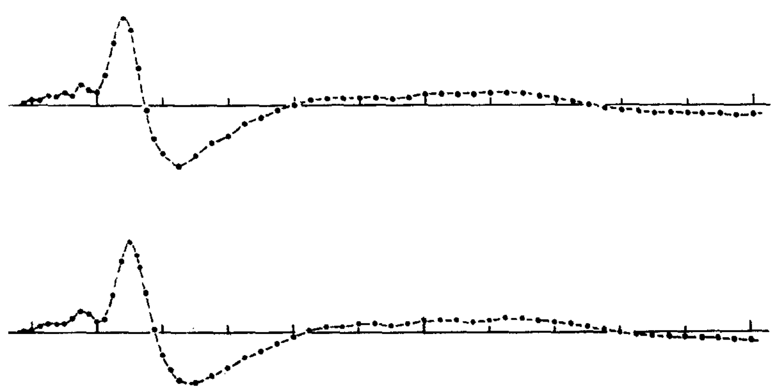

Fig. 11. Curves obtained by the use of the formula (2.7)′. Phase lag L was assumed to be $\pi/4$. For the curve below, the distance r was assumed to be 0.1 cm.

Figure 2 shows the disturbance obtained at points with distances $r = 15$ cm. and $r = 20$ cm. The thickness and other physical constants measured by the experiment are as follows:

$$
\begin{aligned}
H &= \text{thickness} = 1.00\ \text{mm.} \\
\rho &= \text{density} = 2.686\ \text{gr/cm.}^3 \\
V_p &= \text{velocity of P waves} = 5342\ \text{m/sec.} \\
V_s &= \text{velocity of S waves} = 3108\ \text{m/sec.} \\
\sigma &= \text{Poisson's ratio} = 0.244
\end{aligned} \tag{3.1}
$$

The spectra corresponding to the curves in figure 2 are given in figure 3. Mean values of these two curves in the figure were adopted as the function $F(p; r_A)$ in (2.7). See figure 4. The dispersion curve, which was obtained using the equation (2.8), is given in figures 5 (linear scale) and 6 (log-log scale). The latter figure shows that the curve is a straight line with an inclination ½. This is the nature expected for the fundamental mode of flexural wave in a thin elastic plate.[4] From the coefficient of the type of expression

$$V(p) = V_s \cdot \left(\frac{2\pi}{T}\frac{H}{V_s}\right)^{1/2} \cdot \left\{\frac{1}{3}\left(1 - \frac{V_s^2}{V_p^2}\right)\right\}^{1/4} \tag{3.2}$$

we can determine the thickness of this plate, and the value obtained is 0.981 mm.

The velocity and the period of the wave are about 1 km/sec. and 0.01 msec., respectively; therefore the wave length is about 1 cm., which is much longer than the thickness of the plate, whereas compared with the distance of observation points from the origin it is very small. Consequently we can apply the simple law of the propagation of the fundamental mode of flexural waves.

The effect of absorption is small and we omitted the effect given by the factor exp $(pr/2QV)$. Putting the numerical values of $F(p; r_A)$, $V(p)$, and $\beta(p)$, which are given in figure 7, into the expression (2.7), we obtained the curves for $r = 7$ cm. and 3 cm. (figs. 8 and 9). The similarity of the observed and calculated curves is fairly good.

The application of the same formula for the case $r = 0$ cm. gives a rather large discrepancy between the observed and calculated curves (fig. 10). This is the result easily expected, and the formula (2.7)′ was employed, which was corrected by the factor (2.10), assuming

$$L(r_0, r) = \pi/4 \tag{3.4}$$

Further, the computation was carried out assuming $r = 0.1$ cm., and the result seems to be improved a little (fig. 11).

Discussion

In our previous work the calculated curves were not compared with the direct observation. In the present study, however, the comparison was made, and it proved that the calculated curves give fairly good representation of the observed disturbance. However, there is still some discrepancy between two kinds of curves. Calculated maxima and minima come a little earlier than the observed ones. Besides, there is a small disturbance before the time when the disturbance should start. These may be attributed to the following:

1) That too simple a propagation formula was used;

2) That high- and low-frequency parts of the spectrum were cut off for the numerical calculation; and

3) That there may be a small error in the measurement of the distance.

An investigation with better precision, and, too, application to another case, including natural earthquakes, are desirable.

[4] Y. Satô, "Study on Surface Waves, II: Velocity of Surface Waves Propagated upon Elastic Plates," *Bull. Earthq. Res. Inst.*, 29: 223–261 (1951). Errata are given in the paper cited in note 1, first paper.

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